NOTE ON THE NUMERICAL TRANSCENDENTS S_n AND $s_n = S_n - 1$.

BY PROFESSOR W. WOOLSEY JOHNSON.

1. The numbers defined by the series

$$S_n = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots,$$

where n is a positive integer greater than unity, are of frequent occurrence in analysis. Euler, in the Institutiones Calculi Differentialis, 1775, gave a table of their values up to S_{16} to sixteen places of decimals (page 456). He had connected the values of the even-numbered ones with Bernoulli's numbers and the even powers of π by the formula

$$S_{2n} = \frac{2^{2n-1} B_n \pi^{2n}}{(2n)!},$$

but failed to obtain a finite expression for the odd-numbered ones. He gave also the value of the constant γ, now known as Euler's constant, defined as the limiting value

$$\gamma = \left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \log n \right]_{n=\infty}.$$

2. The constants γ and S_n occur in the expressions for the function $\log \Gamma(1+x)$; and Legendre, for the purpose of constructing his table of $\log \Gamma(a)$, examined Euler's table of values for S_n and, finding "quelques erreurs assez graves," reconstructed the table, carrying it to S_{35}, remarking that for higher values of n one has only to divide the excess over unity successively by 2. This is of course because the powers of 1/3, 1/4, etc., have disappeared from the last retained decimal place, which was as in Euler's table the sixteenth.

The values of S_n rapidly approach that of the first term, which is unity. It follows that, an algebraic series being given in which the S_n's occur, if the series resulting from replacing S_n by unity has been already summed, we can by subtraction

obtain a much more rapidly convergent series. Thus Legendre replaced his series

$$\log \Gamma(1+x) = \tfrac{1}{2} \log \frac{\pi x}{\sin \pi x} - \gamma x - \tfrac{1}{3} S_3 x^3 - \tfrac{1}{5} S_5 x^5 - \cdots$$

by

$$\log \Gamma(1+x) = \tfrac{1}{2} \log \frac{\pi x}{\sin \pi x} - \tfrac{1}{2} \log \frac{1+x}{1-x} + (1-\gamma)x - \tfrac{1}{3}(S_3 - 1)x^3 - \tfrac{1}{5}(S_5 - 1)x^5 - \cdots$$

3. I propose in this note to put $S_n = 1 + s_n$, so that

$$s_n = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots.$$

A table of values of S_n is of course virtually a table of values of s_n.

When values of x for which the Γ-function is known are substituted in the algebraic series, numerical series involving S_n and γ are found; thus Legendre * derives from the above equation by putting $x = 1$ and $x = \frac{1}{2}$,

$$(1) \qquad 1 - \gamma = \tfrac{1}{2} \log 2 + \tfrac{1}{3} s_3 + \tfrac{1}{5} s_5 + \tfrac{1}{7} s_7 + \cdots,$$

$$(2) \qquad 1 - \gamma = \log \tfrac{3}{2} + \frac{s_3}{3 \cdot 2^2} + \frac{s_5}{5 \cdot 2^4} + \frac{s_7}{7 \cdot 2^6} + \cdots.$$

Legendre computed γ from each of these series by means of his table of S_n, and cites the agreement with the known value of γ as a test of the accuracy of the table.

Dr. Glaisher in a paper "On the history of Euler's constant," *Messenger of Mathematics*, 1871, cites a number of these relations from the Memoirs of Euler, one of which is, in the present notation,

$$(3) \qquad \gamma = \tfrac{1}{2} s_2 + \tfrac{2}{3} s_3 + \tfrac{3}{4} s_4 + \cdots.$$

This formula was given in 1769, and in a memoir of 1781 occurs the formula

$$(4) \qquad 1 - \gamma = \tfrac{1}{2} s_2 + \tfrac{1}{3} s_3 + \tfrac{1}{4} s_4 + \cdots.$$

* Traité des fonctions elliptiques et des intégrales eulériennes, vol. 2, p. 434. The table of values of S_n is on p. 432.

4. It was a comparison of these equations which first led me to notice the very simple relation, independent of γ,

$$1 = s_2 + s_3 + s_4 + \cdots. \tag{5}$$

This relation must doubtless have been frequently noticed by those who have had occasion to deal with these numbers,* yet it seems clear that it was not known to Legendre writing in 1826, for it forms a much better verification of his table (involving, as it does, every figure of it) than does the computation of γ mentioned above.

5. It is the main object of this note to point out that not only this result but the other results mentioned above involving γ and naperian logarithms may be derived by direct summation of series in s_n's.

Thus, if we write out the terms of each s_n in a column, we have

$$\begin{aligned}\sum_2^\infty s_n = &(\tfrac{1}{2})^2 + (\tfrac{1}{2})^3 + (\tfrac{1}{2})^4 + \cdots \\ &+ (\tfrac{1}{3})^2 + (\tfrac{1}{3})^3 + (\tfrac{1}{3})^4 + \cdots \\ &+ (\tfrac{1}{4})^2 + (\tfrac{1}{4})^3 + \cdots \\ &+ \cdots.\end{aligned}$$

The rows form geometric progressions, hence we have

$$\begin{aligned}\sum_2^\infty s_n &= \frac{(\frac{1}{2})^2}{1-\frac{1}{2}} + \frac{(\frac{1}{3})^2}{1-\frac{1}{3}} + \frac{(\frac{1}{4})^2}{1-\frac{1}{4}} + \cdots \\ &= \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots = 1 - \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{3} + \cdots = 1.\end{aligned}$$

In like manner we can show that

$$s_2 - s_3 + s_4 - \cdots = \tfrac{1}{2};$$

whence it follows that

$$s_2 + s_4 + s_6 + \cdots = \tfrac{3}{4}, \tag{6}$$

* I have, since writing the above, found it given as a problem in the second edition of Boole's Finite Differences in the form: "Shew that the sum of all the negative powers of all whole numbers (unity being in both cases excluded) is unity; if odd powers are excluded, it is $\frac{3}{4}$."

and

$$s_3 + s_5 + s_7 + \cdots = \tfrac{1}{2}. \tag{7}$$

Legendre's table is completely verified by these equations, the sums being in fact only 2 or 3 units out in the 16th place.

Again, to sum $\frac{1}{2}s_2 + \frac{1}{3}s_3 + \cdots$, we have

$$\begin{aligned}\sum_2^\infty \frac{s_n}{n} = {} & \tfrac{1}{2}(\tfrac{1}{2})^2 + \tfrac{1}{3}(\tfrac{1}{2})^3 + \tfrac{1}{4}(\tfrac{1}{2})^4 + \cdots \\ & + \tfrac{1}{2}(\tfrac{1}{3})^2 + \tfrac{1}{3}(\tfrac{1}{3})^3 + \cdots \\ & + \tfrac{1}{2}(\tfrac{1}{4})^2 + \tfrac{1}{3}(\tfrac{1}{4})^3 + \cdots \\ & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\end{aligned}$$

Here each row is of the form

$$\tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \cdots = \log \frac{1}{1-x} - x,$$

where x has the values $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc., in the successive rows. Hence

$$\begin{aligned}\sum_2^\infty \frac{s_n}{n} = {} & \left[\log 2 + \log \tfrac{3}{2} + \log \tfrac{4}{3} + \cdots + \log \frac{m}{m-1} \right. \\ & \left. - \left(\tfrac{1}{2} + \tfrac{1}{3} + \cdots + \frac{1}{m}\right)\right]_{m=\infty} \\ = {} & \left[\log m - \left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \frac{1}{m}\right)\right]_{m=\infty} + 1 = 1 - \gamma,\end{aligned}$$

which is equation (4) above.

6. Treating the sum $s_2 + \frac{1}{2}s_4 + \frac{1}{3}s_6 + \cdots$ in the same manner, the rows are of the form

$$x^2 + \tfrac{1}{2}x^4 + \tfrac{1}{3}x^6 + \cdots = \log \frac{1}{1-x^2},$$

x having the successive values $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, etc. Thus

$$\begin{aligned}\sum_1^\infty \frac{s_{2n}}{n} = {} & \log \tfrac{4}{3} + \log \tfrac{9}{8} + \log \tfrac{16}{15} + \cdots \\ = {} & \log \left[\tfrac{2}{1} \cdot \tfrac{2}{3} \cdot \tfrac{3}{2} \cdot \tfrac{3}{4} \cdot \tfrac{4}{3} \cdot \tfrac{4}{5} \cdots\right] = \log 2.\end{aligned} \tag{8}$$

As a further example, Legendre's equation, (2) of this note, may be verified. Thus in summing $\frac{s_3}{3.2^2} + \frac{s_5}{5.2^4} + \cdots$, the rows take the form, if $y = \frac{1}{2}x$,

$$\frac{x^3}{3 \cdot 2^2} + \frac{x^5}{5 \cdot 2^4} + \frac{x^7}{7 \cdot 2^6} = 2\left[\frac{y^3}{3} + \frac{y^5}{5} + \cdots\right] = \log \frac{1+y}{1-y} - 2y$$

where y takes the successive values $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$, etc. Hence

$$\sum_1^\infty \frac{s_{2n+1}}{(2n+1)2^{2n}} = \log\left[\tfrac{5}{3}\ \tfrac{7}{5}\ \tfrac{9}{7} \cdots \tfrac{2n+1}{2n-1}\right]_{n=\infty} - \left[\tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots + \tfrac{1}{n}\right]_{n=\infty}$$

$$= \left[\log(2n+1) - \log 3 + 1 - (1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{n})\right]_{n=\infty}$$

and, since $\log(2n+1)$ at the limit becomes $\log 2 + \log n$, the result is

$$\sum_1^\infty \frac{s_{2n+1}}{(2n+1)2^{2n}} = \log \tfrac{2}{3} + 1 - \gamma,$$

which is equation (2).

7. I have been tempted to make an independent calculation of the values of s_n, using 12 decimal places. The values of s_n when $n > 9$, when carried thus far, are readily obtained by direct summation of all their terms. But the extreme slowness of the convergence of the terms when n is small renders it practically necessary to employ the same method as that by which γ has been computed, namely the Euler-Maclaurin formula for the summation of a finite number of terms, viz.:

$$\Sigma u_x = C + \int u_x dx - \tfrac{1}{2}u_x + \frac{B_1}{2}\frac{du_x}{dx} - \frac{B_2}{4!}\frac{d^3u_x}{dx^3} + \frac{B_3}{6!}\frac{d^5u_x}{dx^5} - \cdots$$

In the application of this formula when u_x is a negative power of x, the constant C is the sum to infinity, and is obtained by direct summation of Σu_x to a moderate value of x, and calculation of the infinite series in the second member. It was thus that Euler and Legendre calculated the value of γ as well as the values of S_n, and thus that Adams, after greatly extend-

ing the range of known Bernoullian numbers, calculated γ to 263 place of decimals.

Inasmuch as Legendre's table has not often been reprinted, it may be of interest to give the results of my computation to the eleventh place of decimals. They are as follows:

$$\text{Values of } s_n = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots$$

n	s_n	n	s_n
2	.64493 40668 5	14	.00006 12481 4
3	.20205 69031 6	15	3 05882 4
4	8232 32337 1	16	1 52822 6
5	3692 77551 4	17	76372 0
6	1734 30619 8	18	38172 9
7	834 92773 8	19	19082 1
8	407 73562 0	20	9539 6
9	200 83928 3	21	4769 3
10	99 45751 3	22	2384 5
11	49 41886 0	23	1192 2
12	24 60865 5	24	596 1
13	12 27133 5	25	298 0

[The values for $n > 25$ are obtained each by dividing its predecessor by 2.]

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ON CERTAIN PROPERTIES OF WRONSKIANS AND RELATED MATRICES.

BY PROFESSOR D. R. CURTISS.

(Read before the Chicago Section of the American Mathematical Society, April 14, 1906.)

In this note I shall present theorems of a very general character on the vanishing of Wronskians and related matrices. Proofs, however, will be reserved for subsequent publication in more extended form.

Let $u_1, u_2, \cdots, u_n$ be functions, real or complex, of the real variable x, having finite derivatives of the first k orders

($k \geqq n-1$) at each point of an interval I. This interval may be finite or infinite, and, if limited in either direction, may or may not include its end points. We shall use the symbol $M_k(u_1, u_2, \cdots, u_n)$ to designate the matrix

$$\begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ u_1^{(k)} & u_2^{(k)} & \cdots & u_n^{(k)} \end{vmatrix}.$$

The Wronskian $W(u_1, u_2, \cdots, u_n)$ will then be the n-rowed determinant whose matrix is $M_{n-1}(u_1, u_2, \cdots, u_n)$.

The first three of the following theorems concern the vanishing of the Wronskian in an infinite set of points belonging to the interval I, and having a limiting point p which is also a point of I. The symbol $[P]$ will be used to designate such a set when it includes the point p. A set which does not contain the limiting point p will be referred to as a set $[P^*]$.

THEOREM I. *Let $u_1, u_2, \cdots, u_n$ be functions of x which at every point of I have finite derivatives of the first n orders; then if $W(u_1, u_2, \cdots, u_n)$ vanishes in a point set $[P]$ at least one of the Wronskians $W(u_1', u_2', \cdots, u_\nu')$ $(\nu = 1, 2, \cdots, n)$ vanishes in a set $[P^*]$.*

THEOREM II. *Let $u_1, u_2, \cdots, u_n$ be functions of x which at every point of I have continuous derivatives of the first k orders $(k \geqq n)$; then if $W(u_1, u_2, \cdots, u_n)$ vanishes in a point set $[P]$, all the n-rowed determinants of the matrix $M_k(u_1, u_2, \cdots, u_n)$ vanish at the point p.*

In this last theorem the functions $u_1^{(k)}, u_2^{(k)}, \cdots, u_n^{(k)}$ are supposed continuous. The following theorem does not make this hypothesis:

THEOREM III. *Let $u_1^{(k)}, u_2^{(k)}, \cdots, u_n^{(k)} (k \geqq n)$ exist and be finite at every point of I; then if $W(u_1, u_2 \cdots, u_n)$ and its first $k - n + 1$ derivatives vanish simultaneously in a point set $[P]$ all the n-rowed determinants of the matrix $M_k(u_1, u_2, \cdots, u_n)$ vanish simultaneously either at p or in a set $[P^*]$.*

From these three theorems we deduce the following, which is easily seen to be a direct consequence of Theorem II when $u_1^{(k)}, u_2^{(k)}, \cdots, u_n^{(k)}$ are continuous:

THEOREM IV. *Let $u_1^{(k)}, u_2^{(k)}, \cdots, u_n^{(k)} (k \geqq n)$ exist and be finite at every point of I; then if $W(u_1, u_2 \cdots, u_n)$ vanishes identically throughout I every n-rowed determinant of $M_k(u_1, u_2, \cdots, u_n)$ vanishes identically throughout I.*

For an interval where $W(u_1, u_2, \cdots, u_{n-1})$ does not vanish, Theorem IV has been stated and proved by Bôcher*; this restriction is not necessary to my proof. As a corollary we obtain Bôcher's theorem that if $W(u_1, u_2, \cdots, u_n) \equiv 0$ then $W(u_1, u_2, \cdots, u_{n+1}) \equiv 0$, but Bôcher's assumption of the continuity of $u_1^{(n)}, u_2^{(n)}, \cdots, u_{n+1}^{(n)}$ is now clearly seen to be unnecessary.

The preceding theorems have an important application in the theory of linear dependence. The results which I have obtained are summarized in the following theorem where the phrase, "$M_k(u_1, u_2, \cdots, u_n)$ is of constant rank $m < n$ in I," means that every $(m+1)$-rowed determinant of this matrix vanishes identically throughout I while the m-rowed determinants do not all vanish at any point of I:

THEOREM V. *If $M_k(u_1, u_2, \cdots, u_n)$ is of constant rank $m < n$ in I, $u_1, u_2, \cdots, u_n$ are linearly dependent and the number of independent linear relations between these functions is $n - m$.*

Since $M_k(u_1, u_2, \cdots, u_n)$ is of constant rank if $u_1, u_2, \cdots, u_n$ are solutions of a linear differential equation of order k, it is evident that Theorem V is of wide application. In fact it includes all sufficient conditions for the linear dependence of non-analytic functions that have been given by Peano and Bôcher, but applies to no cases which do not come under Bôcher's Theorem VI.† This theorem states that if $W(u_1, u_2, \cdots, u_n) \equiv 0$, while no function (other than zero) of the form

$$g_1 u_1 + g_2 u_2 + \cdots + g_n u_n$$

(the g's being constants) vanishes together with its first k derivatives at any point of I, the functions $u_1, u_2, \ldots, u_n$ are linearly

* *Transactions Amer. Math. Society*, vol. 2 (1901), p. 139. In this article the author gives references to other papers on Wronskians by Peano and himself.

† *Loc. cit.*, p. 144.

dependent. With the aid of Theorem V it can be shown that a necessary and sufficient condition for the verification of the hypothesis of Bôcher's theorem is that $M_k(u_1, u_2, \cdots, u_n)$ be of constant rank $m < n$.

NORTHWESTERN UNIVERSITY,
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SIGNIFICANCE OF THE TERM HYPERCOMPLEX NUMBER.

BY PROFESSOR JAMES BYRNIE SHAW.

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ESSENTIALLY four definitions of quite different logical import have been given for the term hypercomplex number, or multiple number. The four things so defined differ considerably in their mathematico-philosophical meaning, and while two of them are in a way equivalent, neither of the others can be correlated with these two or with each other as equivalent. It is proposed to examine these four definitions rather critically.

I. *The n-tuple.*

The first definition I shall denominate the Dedekind definition, although Hamilton discussed, many years before, entities defined in the same way. It is of a pure arithmetical character, since it implies only the existence of a set of things we may call numbers, marks, or entities, according as we conceive them to belong to a domain of integrity, an abstract field, or, in general, an aggregate that we can call a range. At first these entities were in a scalar domain, then they were generalized to a rational domain, then to an abstract field, and obviously we may take them from any range. The definition runs substantially thus: *

A set of n ordered marks (entities) $a_1, \cdots, a_n$ of a field (range) F, is called an n-tuple element a. The symbol $a = (a_1, \cdots, a_n)$ employed is purely positional without functional connotation. Its definition implies that $a = b$ if, and only if, $a_1 = b_1, \cdots, a_n = b_n$.

* Dickson, "On hypercomplex number systems"; *Transactions Amer. Math. Society*, vol. 6 (1905), pp. 344-348.

This system of n-tuple elements is then made to be linear and associative by a set of postulates of such a nature that the sum and the product of any two elements is a third element. Like all arithmetic definitions, this definition has the merit of being free from existential questions. If marks of a field (or entities of a range) exist, these elements exist. The element is a conception which is a function of its n marks, or n coördinates. The conception is one needed in many applications of mathematics, hence its introduction is justified. But logically these elements are not extensions of the ordinary numbers and can be so called only by an extension of the meaning of the term number, and this extension must be in an altogether artificial manner. As Bertrand Russell points out with regard to the elements defined in the next definition, we cannot identify * unity with any of the n-tuple elements, not even with the modulus, which may be written

$$(1, 0, 0, \cdots, 0).$$

For unity is not n-tuple, not even an n-tuple with $n - 1$ zero coördinates. The distinction is analogous to that which insists that unity and positive unity are two different conceptions.

The n-tuple has no reference to anything else. It is a compound conception built up directly from the range given. Its laws of combination are purely arbitrary, since there is no reason à priori why, in $(a, b)(x, y) \equiv (\gamma_{111}ax + \gamma_{121}ay + \gamma_{211}bx + \gamma_{221}by, \gamma_{112}ax + \gamma_{122}ay + \gamma_{212}bx + \gamma_{222}by)$, the eight γ's should have one value rather than another. The thing to do, on this basis, obviously is to study all possible types of algebras of such n-tuples, and find their uses, if any.

II. *The Manifold.*

The second definition I shall call the Russell definition, although it has appeared before in works of other logicians. It runs thus:

What defines a hypercomplex number is a one-many relation whose domain consists of real numbers (that is, ordinary rationals, irrationals, etc.), and whose converse domain consists of the first n integers (or, in the case of complex numbers of in-

* Principles of Mathematics, vol. 1, p. 380.

finite order, of all integers). These relations are indicated by $e_1, e_2, \cdots, e_n$. Thus, in $a = a_1e_1 + a_2e_2 + \cdots + a_ne_n$, a_1 is correlated to 1, a_2 to 2, $\cdots$, a_n to n. The relation is one-many because a_i may equal a given number b, for $i = 1, 2, \cdots, n$; that is, b is correlated to each integer; so for every number. These one-many relations may be defined to be the complex numbers. In this way a purely logical definition is obtained. We are not to identify $1\,e_1 + 0e_2 + \cdots + 0e_n$ with e_1, that is, the complex number is a function of the coordinates, not of the relations e. Evidently we may use any other range than that of all real numbers, and this definition becomes as general as the first, which it most resembles. Russell proceeds immediately to show that this virtually defines a complex number to be an n-dimensional entity. It is substantially the definition at the foundation of vector analysis, where the relational symbols $e_1, e_2, \cdots, e_n$ are expressed by directions.

It is sufficiently clear that these manifolds are not in the ordinary number system any more than were the n-tuples.

Their sums and their products are likewise manifolds whose coordinates are determined by certain laws of combination. True, we may call them numbers, but the identification of an ordinary number with any one of the manifolds is highly artificial. In fact, students of space analysis or vector analysis would scarcely conceive of any such identification being possible, any more than they would think of identifying an angle with its number of degrees.

The connection of the n-tuple and the manifold with the bilinear function of n variables is evident.

This is the point of view of the qualitative-unit mathematician, a unit 1 correlated to the n integers gives the n units $1e_1, 1e_2, \cdots, 1e_n$, of the system. These are usually written $e_1, \cdots, e_n$, although $1e_1$ is not really e_1; for e_1 really expresses only that something is correlated to 1. The correlation may as well be with n directions in space, and might be written $i, j, k, \cdots$. In this case the qualitative phase is a little more apparent. Of course we are left free to say what $ae_1 \times be_2$ shall mean; this gives rise to a multiplication table, meaning that if a is correlated to 1, b to 2, then ab shall be correlated to $\gamma_{121}e_1 + \gamma_{122}e_2 + \cdots + \gamma_{12n}e_n$. This view appeals strongly to the quaternionist or vector analyst who desires a flexible symbolism, although many use the cumbrous n-tuple forms.

III. *The Operator.*

The third definition I shall call the Peano definition, although it appeared long before in Hamilton's quaternions. This definition looks upon the hypercomplex number as an operator. What Peano* calls a complex number is incapable of multiplication, as he explicitly says, hence it is not the kind of entity we are discussing here. But he does not include quaternions, $\sqrt{-1}$, and the like, in the class of complex numbers. These he defines as operators. Thus, $\sqrt{-1}$ is the substitution $\left|\begin{smallmatrix}0 & -1\\ +1 & 0\end{smallmatrix}\right|$, that is, if we have given a non-multiplicative manifold (x, y), then by $\sqrt{-1}\ (x, y)$ we mean $(-y, x)$. Obviously matrices come under this head, as do linear substitutions.

It will readily be granted, I suppose, that these operators are not numbers, although Hamilton identifies quaternions that are purely scalars with the domain of positive and negative numbers. But he means that the scalar 1 is an operator such that applied to a, any vector, $1\,a = a$. If we were to define numbers as operators, then, of course, scalars are numbers. But numbers defined on a pure arithmetical basis are not operators.

It is to be noted carefully in these three definitions that the product or sum of any two elements is a third element of the same character. No possible combination ever leads us away from the range under consideration. In the first case we have, for example, the products of two n-tuples: $(a, b)(x, y) = (ax\gamma_{111} + ay\gamma_{112} + bx\gamma_{121} + by\gamma_{122}, ax\gamma_{211} + ay\gamma_{212} + bx\gamma_{221} + by\gamma_{222})$, an n-tuple. To connect these with numbers, we virtually must define real numbers to be n-tuples. In the second case we have for the product of two manifolds $(ae_1 + be_2)(xe_1 + ye_2) = [(ax\gamma_{111} + ay\gamma_{112} + bx\gamma_{121} + by\gamma_{122})\,e_1 + (ax\gamma_{211} + ay\gamma_{212} + bx\gamma_{221} + by\gamma_{222})e_2]$, a manifold. To connect with numbers, we must define numbers as manifolds. In the third case we have for the product of two operators $(ai_1 + bi_2)(xi_1 + yi_2)(\ ) = [ax\gamma_{111} + ay\gamma_{112} + bx\gamma_{121} + by\gamma_{122})\,i_1 + (ax\gamma_{211} + ay\gamma_{212} + bx\gamma_{221} + by\gamma_{222})\,i_2](\ )$, an operator. To connect with numbers, we must define a number as an operator.

**Formulaire* (1901).

IV. *The Congruence.*

The fourth definition I shall call the *algebraic* definition. It is so called by Russell, who says indeed regarding this method * "... the algebraic generalisation is very inferior to the arithmetical, since the latter gives all irrationals by a uniform method, whereas the former, strictly speaking, will give only the algebraic numbers. But with regard to the complex numbers, the matter is otherwise. No arithmetical problem leads to these, and they are wholly incapable of arithmetical definition. But the attempt to solve such equations as $x^2 + 1 = 0$, $x^2 + x + 1 = 0$ at once demands a new class of numbers, since, in the whole domain of real numbers, none can be found to satisfy these equations. To meet such cases, the algebraic generalisation defined new numbers by means of the equations whose roots they were. ... But the algebraic generalisation was wholly unable (as it was, in truth, at every previous stage) to prove that there are such entities as those which it postulated."

Without agreeing or disagreeing with this statement as it stands, we may place the algebraic definition on the following unimpeachable basis: † Let us consider that i is such an entity that wherever we find the expression $i^2 + 1$ we may write 0 for it; that is, we reduce all our problems, as it were, modulo $i^2 + 1$. Then we may write $x^2 + x + 1 = x^2 + x + \frac{1}{4} + \frac{3}{4} = x^2 + x + \frac{1}{4} + \frac{3}{4} - \frac{3}{4}(i^2 + 1) = x^2 + x + \frac{1}{4} - \frac{3}{4} i^2$. This evidently has the factors $x + \frac{1}{2} - \frac{1}{2}\sqrt{3}\, i$ and $x + \frac{1}{2} + \frac{1}{2}\sqrt{3}\, i$. Hence we may speak of the roots as $-\frac{1}{2} + \frac{1}{2}\sqrt{3}\, i$ and $-\frac{1}{2} - \frac{1}{2}\sqrt{3}\, i$. We have not made any assertions at all regarding the nature of i, and may if we choose look upon it as simply a perfectly arbitrary number, or algebraic *form* destitute of numerical value. We may look upon all radicals in the same way; thus $x^2 + 6x + 7$ has no rational roots. But if we work modulo $j^2 - 2$, we find at once that we may write $x^2 + 6x + 7 = x^2 + 6x + 9 - j^2$, and we say $x + 3 \pm j = 0$. We call j an extension of our number system. So we call i an extension of the number system.

* Principles of Mathematics, p. 377 et seq.

† With regard to this basis, I desire to acknowledge valuable suggestion from Professors Moore and Dickson.

If now we go farther and work modulo $i^2+1, j^2+1, k^2+1, l^2+1, \cdots$, where $i, j, k, l, \cdots$ are different forms, we find that we may take $(i+j)^2+2=0$ if $ij+ji=0$. Hence it is suggested that we must either consider $ij=0$, or else consider that ij and ji are not equal. That is, we must choose between nil-factors with commutativity, or else give up commutativity. We do the latter and arrive at quaternions. So that we have expressions reducible modulo

$$i_1^2+1,\quad j_1^2+1,\quad k_1^2+1,\quad i_2^2+1,\quad j_2^2+1,\quad k_2^2+1,\cdots,$$

$$i_1j_1+j_1i_1,\quad j_1k_1+k_1j_1,\quad k_1i_1+i_1k_1,\quad i_2j_2+j_2i_2,\cdots,$$

$$i_1j_2,\quad i_2j_1,\quad i_1k_2,\quad i_2k_1,\quad j_1k_2,\quad j_2k_1,\quad j_1i_2,\quad j_2i_1,$$
$$k_1i_2,\quad k_2i_1,\quad k_1j_2,\quad k_2j_1,\quad \cdots.$$

If we use only i_1, j_1, k_1, we find we may factor the expression

$$x^2-2ax+a^2+b^2+c^2+d^2,$$

namely

$$x^2-2ax+a^2+b^2+c^2+d^2=(x-a+bi_1+cj_1+dk_1)$$
$$(x-a-bi_1-cj_1-dk_1).$$

We have thus arrived at quaternions. Here again we may consider i_1, j_1, k_1 as merely formal, or we may look on them as extensions of the number system. In the end the latter view seems more profitable.

We may then state the algebraic definition for the general case as follows: Let there be an expression, or set of expressions, in terms of $q, r, \cdots$ such that we reduce all expressions in terms of the entities of a given range, modulo these given expressions, where $q, r, \cdots$ represent any entities for which this can be done. Then the entities q, say $q_1, q_2, \cdots$, and the entities r, say $r_1, r_2, \cdots$, etc., define extensions of the given range. If the number of these ideal entities is finite we arrive at an algebra of a finite number of dimensions, otherwise an algebra of infinite dimensions.

V. *Conclusions.*

This brings us to one of the objects of the whole discussion. Let me put it in the form of a simple question. Do positive and negative numbers constitute a complex number system of two units $e_0=+1$, $e_1=-1$, with the domain of ordinary

arithmetical rational and irrational numbers for the coordinate range, or do they not constitute a complex system, and, if not, why not? Surely $+1$ and -1 are units and are qualitatively distinct. According to the second definition, then, we may define $a_1(+1)+a_2(-1)$ as a complex number of two units. But in this algebra we know that $a_1(+1)+a_2(-1)=(a_1-a_2)(+1)$ when $a_1>a_2$; $=0$, when $a_1=a_2$; and $=(a_2-a_1)(-1)$ when $a_2>a_1$. Hence these qualitative units are such that though themselves distinct, when coordinates are attached, we arrive at expressions which are not unique. Thus if $(a_1, a_2)=(b_1, b_2)$, we may infer $a_1=b_1, a_2=b_2$; and if $(a_1e_1+a_2e_2)=(b_1e_1+b_2e_2)$, then $a_1=b_1$, $a_2=b_2$. But if $a_1(+1)+a_2(-1)=b_1(+1)+b_2(-1)$, the most we can infer is that $a_1=b_1\pm b$, $a_2=b_2\pm b$. We meet the same problem in operators and simply evade it by defining $(+1)\alpha+(-1)\alpha=0\alpha$. The question still remains.

If an n-tuple may reduce to an $(n-m)$-tuple it ceases to be properly an n-tuple. Hence the n-tuple algebra must exclude the positive-and-negative-number calculus as not an algebra. So also if the n-manifold may have its number of dimensions reduced, it ceases to be an n-manifold. So this calculus must be excluded here. A vector algebra of positive and negative quantities is one-dimensional and not properly a vector algebra at all. In fact, the n-tuple and the manifold exclude any possibility of ever reducing the number of positions or of units by additions of their coordinates which belong to different positions or different units. Technically, the units are linearly independent so far as entities of the range are concerned. Now $+1$, -1, are not linearly independent so far as the domain of the integer or the real number are concerned. Thus, for example $2(+1)+1(-1)=1(+1)$.

The third definition places the difficulty farther back. Thus $(+1)$ and (-1) are independent or not according as $(+1)\,a$ and $(-1)\,a$, where a is the operand, are independent or not.

The fourth definition admits such lack of independence, for we define -1 by this very lack of independence. Indeed, we define -1 to be the entity x such that

$$x+1=0 \quad \text{or} \quad x+a=a-1.$$

That is, we work modulo $x+1$. We extend our old arithmetical addition to a new ideal range. On the basis of the

fourth definition, positive and negative numbers do constitute an algebra of two distinct qualitative units, but not a two-dimensional algebra. In fact, we may assert at once that the fourth definition defines ideal numbers which are qualitatively distinct, but which may not be able to be ordered as many-dimensional. We do not at first know whether the different ideals are many-dimensional or not. Thus $q^2 - 2wq + t^2 = 0$ is satisfied by any number of sets of quaternions, and the dimensions may thus become infinite.

The distinction may be illustrated by the difference between a line in a two-dimensional bounded non-euclidean space and such space itself. The portion of the line external to the space may be called ideal, but the real and the ideal (qualitatively different) portions of the line are yet one-dimensional.

Modifications of the usual theory of algebras are necessary in these algebras with, we may say, deficiency of dimension. Thus for positive and negative numbers we do not have a characteristic equation of order two, which we should expect, but simply the equation

$$(+\ 1) + (-\ 1) = 0.$$

Another example occurs in ordinary complex numbers, which constitute an algebra of four qualitative units, e_0, e_1, e_1^2, e_1^3, with the equations $e_0 + e_1^2 = 0 = e_1 + e_1^3$. The characteristic equation is not of order four, but for $\rho = xe_0 + ye_1 + ze_1^2 + we_1^3$ is $(\rho - x + z)^2 + (y - w)^2 = 0$.

As another example we might define an algebra by working modulo $e_1^2 - 1$, $e_2^3 - 1$, $e_1e_2 - e_2^2e_1$, $e_2 + e_2^2 + 1$, $e_1e_2 + e_1e_2^2 + e_1$.

These examples show the great power and generality of the fourth definition. It is evident that we define an algebra thus by what may be called its invariant equations.

The criticism made by Russell, that algebraic definitions do not guarantee the existence of the thing defined, does not apply to this method of putting the definition. If algebra exists, then we may work modulo $f(q)$ if we choose, and $f(q)$ is simply an expression in the algebra. We thus arrive at expressions containing q. If we choose to call these hypercomplex numbers, we do not thereby destroy them. If under given conditions it turns out that we may work modulo a whole infinity of expressions $f(q_1)$, $f(q_2)$, ..., then we have simply found a kingdom where we searched for much less.

April, 1906.

HOW SHOULD THE COLLEGE TEACH ANALYTIC GEOMETRY?*

BY PROFESSOR HENRY S. WHITE.

In most American colleges, analytic geometry is an elective study. This fact, and the underlying causes of this fact, explain the lack of uniformity in content or method of teaching this subject in different institutions. Of many widely divergent types of courses offered, a few are likely to survive, each because it is best fitted for some one purpose. It is here my purpose to advocate for the college of liberal arts a course that shall draw more largely from projective geometry than do most of our recent college text-books. This involves a restriction of the tendency, now quite prevalent, to devote much attention at the outset to miscellaneous graphs of purely statistical nature or of physical significance. As to the subject matter used in a first semester, one may formulate the question: Shall we teach in one semester a few facts about a wide variety of curves, or a wide variety of propositions about conics?

Of these alternatives the latter is to be preferred, for the educational value of a subject is found less in its extension than in its intension; less in the multiplicity of its parts than in their unification through a few fundamental or climactic principles. Some teachers advocate the study of a wider variety of curves, either for the sake of correlation with physical sciences, or in order to emphasize what they term the analytic method. As to the first, there is a very evident danger of dissipating the student's energy; while to the second it may be replied that there is no one general method which can be taught, but many particular and ingenious methods for special problems, whose resemblances may be understood only when the problems have been mastered. No one can lay down rules *a priori* which would enable the student to rediscover the theorems now known concerning conics — if these were once lost. Were there such a single well-defined method or art of creating geometry, should we still be waiting for the completion of the theory of plane quartics, or for even an outline of the theory of quintics? It is rather the art of asking questions

* Read before the Association of Teachers of Mathematics of the Middle States and Maryland, at Annapolis, Md., December 1, 1905.

that is to be taught. How can one learn what kind of questions will prove fruitful, save by examining those which have been asked and answered by geometricians in the past?

If the hundreds of important theorems in conics were really unrelated, the argument for their study simply as pertaining to conics would be weak. But just there lies the point; modern geometry has found general principles which unify the whole subject. Every part is connected with every other part. The student learns to employ many methods of inquiry or of proof, but does not lose himself in endless diversity. And while no one can affirm that a branch of science is complete, yet all historians point to the theory of conics as a model both in the large number of its particular propositions and in the possession of general concepts and theorems. It is the most optimistic dream of workers in other branches of mathematics that their favorite special subject may some day reach a stage of development equally advanced with that to which two thousand years have brought this ancient favorite of geometricians.

Conics is an ample range for one semester, and needs but little enlargement for a second. Of the methods to be employed not much need be said to experienced teachers. For the first two weeks, constant drill work on points, distances, areas, circles, and parabolas. By that time the notion of a language for definite use will impress itself. The desire to express a distance in rectangular coördinates calls up a radical covering the sum of two squares, and conversely the sum of two squares of differences of like coördinates recalls involuntarily the motion of a distance; the equation of a circle comes to look to the eye like the definition of a circle in words. A parabola is thought of when an equation separates into one linear part and one square of a different linear expression. Proportional division becomes as familiar as the inflection of a Latin noun, and the area of a triangle in terms of vertices, a string of six terms, comes to look like a moderately long German word. Analysis and synthesis have begun to be instinctive, as they are when one has taken the first steps in a foreign language.

Next comes the patient extension of the vocabulary simultaneously with the study of new geometrical notions. Slope, intercepts, systems of lines, pencils of conics prove easy one after another, and the first milestone is reached when after twenty hours of class-room work the student perceives that

every linear expression denotes a point-function with niveau lines straight and parallel, and that all such expressions may serve as coördinates equally well with his favorite x and y.

This is the point from which geometry should be the main object, and the properties of conics should be developed. Of course metrical properties must come first. The parabola with its axis, focus, vertex, and directrix must be fixed by drawing practice; the bisection of the sub-tangent by the vertex will naturally be the first theorem, and the constancy of the sub-normal the next. Mutually perpendicular tangents meeting on the directrix, and circles circumscribed to circumscribed triangles are important enough to demand attention from even the most hurried student. The diameters of parabolas and of all other conics should be studied together, and conjugate pairs of diameters with their use as axes of reference.

So far, probably every teacher will agree, the student's attention should be confined to metric properties; and indeed in view of the number, the beauty, the practical value of metric properties of conics, it is not surprising that most American text-books include nothing more general. But it is to be remembered that the aspect of this large body of truths to the beginner is not the same as to the more advanced student or teacher. The former is first pleased with the novelty, then wearied with the multiplicity of disconnected facts, and will soon become bewildered and discouraged. A passage from Taylor's Ancient and modern geometry of conics puts the case so well that I will quote it here. "The mind of the tyro is commonly overwhelmed with a multitude of details not reduced to any system, demonstrations are put before him in an unsuggestive form which gives no play to his inventive faculty; and thus it comes to pass that of the many students so few turn out genuine geometers. Let the learner be furnished with principles, and not alone with fully explained facts, and be continually stimulated to exertion by the intense pleasure of finding something left to discover for himself." (P. LXXVII, § 4.) In contrast to this attitude of the neophyte confronted with the wealth of existing theorems of metric character, let any teacher, familiar with projective geometry, review Pascal's chapter (Repertorium der höheren Mathematik, part II) or any other brief synopsis of this subject. He will note the rising tide of pleasure and admiration as well-known and less known discoveries succeed one another, in all some sixty or seventy facts

of major importance; and will probably observe that half of his satisfaction is referable to his serene conviction that all these things flow as easy deductions from a very small number of projective theorems.

Projective geometry is acknowledged to furnish a fascinating continuation or climax to metric geometry; but if it can also serve as a clue, a map amid the intricacies of the elementary theories, why should it not be employed for that purpose? After working two-thirds of a semester to acquire an understanding of the objects and a preliminary knowledge of the theorems of metric conics, let the student learn the few fundamental and comprehensive projective theorems. First of all comes, of course, the relation of conjugate points, commonly stated in the form: If a point P lies on the polar of a point Q, then conversely Q will lie on the polar of P. Without much assistance the student can see that this theorem contains no reference to axes of coordinates, and will suspect that it points to a more general geometry than he has yet studied. Next he can be made acquainted with the harmonic position of four points, or with the anharmonic ratio of any four collinear points or copunctual lines, and then return for practice to the theorems on conjugate diameters. Then he should learn that the anharmonic aspect of four points on a conic is the same from all points of that conic, and the dual theorem on tangents. He has now in hand the projective (organic) generation of the general conic, the unifier of many facts previously independent.

Returning to the metric basis, I would then establish the use of trilinear coordinates by the use of the formula for distance of point from line. Thereupon one may present Salmon's proofs for Carnot's theorem concerning the segments cut by a conic on the sides of a triangle; and if the theorem of Menelaus on the straight transversal of a triangle has been introduced earlier, Pascal's hexagon may be made the terminus of the first semester's study. Less than this ought not to satisfy us. As no college student of chemistry ought to be a whole semester in ignorance of Mendeléeff's table; as no college student of physics should pass an examination before learning, however crudely, some reasons for believing in the conservation of energy; so in analytic geometry let us give even the beginner in his first course some knowledge of Pascal's immortal theorem.

In the college course, the average student of sciences cannot afford time for more than one semester of geometry, even though the analytic auxiliaries may seem to open a whole new world to his vision. He must hasten to master the elements of the differential and integral calculus and attack applied mathematics. Hence the necessity of making the first semester of analytic geometry less slow and richer in material than is at present customary. But provision ought to be made for another year's work for those who wish to become mathematicians. Their second semester should make them acquainted first with quadric surfaces, both as individual surfaces and as associated in a pencil and in a confocal system, but next and by no means less in importance, with the remaining chief propositions concerning plane conics. These are perhaps those of Desargues and Steiner on two perspective triangles, that of Brianchon on the sextette of tangents, with modifications for quintettes, quartettes, and triads; Hesse's famous theorem on the six vertices of a complete quadrilateral, four of which form two conjugate pairs in a polarity, and the theorems of Poncelet concerning polygonal lines inscribed to one conic and circumscribed to another. And lack of famous names should not cause us to omit those relating to pairs of triangles, the first where the triangles are polar reciprocals with respect to a conic, and the second where each is self-polar.

These latter theorems, with due attention to the powerful methods of projection and reciprocation, may well enough fill up the second semester; of course each one should be the occasion of two or three careful exercises in drawing and in schematic constructions. The third semester will find its most appropriate material in mutually apolar systems of conics, and in a rapid discussion of plane cubic curves. At present this is, so to speak, the end of improved roads in one direction, and students who come so far may easily obtain such outlooks as the ambitious are sure to desire; and they will have sufficient experience of travel to warrant them in undertaking explorations upon their own account. Apropos of this we may notice Loria's summary of the present situation, in his valuable compend of Special plane curves (pages 219–20).

"The theory of curves of the third order has clearly become, so to speak, a province of the realm of mathematics; the theory of curves of the fourth order is a territory only partly subjected to this kingdom; but the theory of curves of any

particular order higher than the fourth still asserts its independence and resists stubbornly every attempt to introduce law and order. If we except [certain special researches upon quintic curves] we may say that up to the present nothing has occurred to illuminate the thick darkness which conceals from us the properties of those curves which should come next after conics, cubics and quartics. Not only so, but there is not even any special curve of fifth order which has received a particular name."

The course here advocated is somewhat modernized from the traditional type of college courses in geometry, but it is not, therefore, chimerical. A survey of the field should convince us that in three semesters a college student can become posted on the essentials of conics, quadric surfaces, and cubic curves. Individual teachers will prefer a different order from that here advocated, either for theoretical reasons or on account of the special qualifications of the students under their charge; but it is believed that the main thesis will command general approval — the most modern methods and the most general propositions of present day geometry must be made available for the college student.

VASSAR COLLEGE,
November 29, 1905.

FOUR BOOKS ON THE CALCULUS.

Die Anfangsgründe der Differentialrechnung und Integralrechnung. Für Schüler von höheren Lehranstalten und Fachschulen sowie zum Selbstunterricht. Dargestellt von DR. RICHARD SCHRÖDER. With numerous exercises and 27 figures in the text. Leipzig, B. G. Teubner, 1905. vii + 131 pp.

Hauptsätze der Differential- und Integralrechnung, als Leitfaden zum Gebrauch bei Vorlesungen. Zusammengestellt von DR. ROBERT FRICKE. Fourth edition, with 74 figures in the text. Braunschweig, Friedrich Vieweg und Sohn. xv + 217 pp.

Repetitorium und Aufgabensammlung. Von DR. FRIEDRICH JUNKER. Second, improved edition, with 46 figures in the text. Leipzig, G. J. Göschen, 1905. 16mo. 127 pp.

Sammlung von Formeln und Sätzen aus dem Gebiete der elliptischen Functionen, nebst Anwendungen. Von J. THOMAE. Leipzig, B. G. Teubner. 43 pp.

THESE four titles are here grouped for the purpose of making certain comparisons both among themselves and with current American texts in the corresponding fields.

The first volume, as its title indicates, is an introduction to the calculus and is intended for use in the upper classes of gymnasia, in accordance with the permission afforded in the Prussian curriculum of 1901. While the text is strictly elementary, it is clear and logical in its development, beginning with limits notions and geometric interpretation of the derivative. The derivatives of all ordinary forms, including inverse and implicit functions, are deduced before any exercises are introduced; then over one hundred exercises, very many of which are complicated combinations of algebraic and transcendental functions, are given in one list, with the results in each case printed in juxtaposition. Eleven pages, or one-twelfth of the book, are devoted to indeterminate forms, with illustrations but no exercises. Thirty-three pages are given to tangents, normals, subtangents, subnormals, etc., the forms of which are all worked out for the circle, parabola, ellipse, hyperbola, semicubical parabola, cissoid, cycloid, lemniscate and cardioid, with no exercises left for the student. Likewise rectifications and quadratures for these curves are worked out in full in the text.

The second volume is the fourth edition of Fricke's well-known text, improved and enlarged. When we consider the clearness and rigor of presentation and contemplate the number of subjects here treated within a little more than two hundred pages, including all the ordinary topics of elementary calculus, together with chapters on the theory of definite integrals, ordinary differential equations, and functions of a complex variable, we wonder how it can be done in so small a space. The explanation is that the book contains very few exercises for the student and the development of principles goes on uninterruptedly from beginning to end. From the standpoint of the purely theoretical science, this presentation of the subject has its advantage in that the attention is not diverted from the steady swing of the doctrine from topic to topic; and the student whose mind quickly and readily grasps the principles of the calculus in the abstract must

surely find great delight in the clear and vigorous presentation, the pleasing arrangement of the page, and the skillful development step by step, from the elementary to the more involved theorems. But the student who must handle the concrete before he grasps the abstract, who must work through many exercises up to the comprehension of the principles, will need to combine with such a text as Fricke's a good collection of exercises and applications in the differential and integral calculus.

Fortunately such exercise books are not wanting in Germany, and the third title in the list above is a book of this kind for the differential calculus. It is a small volume containing nearly five hundred exercises, more than half of which are set forms for differentiation or for the formal application of the rules under the various topics, and in all cases the results are given with the exercises, leaving no chance for independent discovery. Of the remaining exercises which are more in the nature of problems, all are geometric, relating to maxima and minima, asymptotes, tangents, normals and numerous properties of curves and surfaces, and under all the topics large numbers of illustrations are worked out in full. There are no application to mechanics or other subjects.

These two volumes of Fricke and Junker (with a corresponding exercise book on the integral calculus) would seem to constitute a somewhat typical provision for the needs of the German student of the calculus, quite in contrast with the plan of our own text-books — on this and other subjects. In favor of the former plan is the greater compactness of the text and the more closely knit continuity in the development of the principles; while the latter plan secures the juxtaposition of principles and applications in a single volume and seems more certain to secure careful attention to the concrete side of the work on the part of the student.

The last title here under consideration represents still another scheme of arrangement and presentation of formulas and principles of a subject, quite in contrast with the other three volumes and also with publications in this country. Thomae has collected the chief formulas and theorems in the realm of elliptic functions, without proofs or explanations beyond what is necessary for a clear understanding of the notation. These are included compactly but clearly under forty-two divisions

covering only twenty-six pages and the remaining eighteen pages are devoted to brief expositions of twenty typical applications of the elliptic functions. The collection of formulas is based on the Jacobi-Legendre fundamental functions and notation, as being, in the author's judgment, better adapted to applications involving numerical computation than the Weierstrassian forms used by Schwarz in his collection, though these latter are of the highest theoretical importance.

Thomae's collection puts in concise and accessible form the whole range of the elliptic function doctrine, as based on the theta function, the zeta function, the omega function, and the Legendre normal forms, and shows fully how useful and practical this development becomes in application to a wide class of problems.

In conclusion it seems quite clear that collections of principles and formulas are highly appropriate and useful for students in advanced stages of progress, but that for elementary students the form of presentation to be commended is that in which problems and exercises are skilfully used in the text to lead up to the statement and proof of principles, as well as to illustrate and clarify the theory in immediate connection with its formal development.

H. E. SLAUGHT.

THE UNIVERSITY OF CHICAGO,
May 28, 1906.

SHORTER NOTICES.

Einleitung in die Funktionentheorie. Abteilung 2. By OTTO STOLZ and J. ANTON GMEINER. Leipzig, B. G. Teubner, 1905. viii + 355 pp.

WITH the publication of the second part of the Stolz and Gmeiner Funktionentheorie, the revision of Stolz's Allgemeine Arithmetik is complete. The Theoretische Arithmetik and the Funktionentheorie, which must still be regarded as parts of the same whole, together present a course in analysis which begins with the integers and includes all the usual operations except differentiation and integration. Taken in connection with Stolz's Calculus, they form a kind of German Cours d'Analyse.

The volume before us begins with Chapter VI, containing a thorough discussion of convergence and divergence criteria for infinite series. The chapter is new and, according to the preface, due to Gmeiner, who has based it principally on the article of Pringsheim in the thirty-fifth volume of the *Mathematische Annalen*. The subject is an extremely interesting one, connecting with several of the most difficult problems in analysis. Its connection with the related problem of determining scales to represent the orders of infinity of functions is fairly evident though not emphasized; but its equivalence with the problem of the convergence of

$$\int_a^\infty f(x)dx = \mathop{\mathrm{L}}_{b=\infty} \int_a^b f(x)dx$$

we did not find referred to, doubtless because the book does not take up the theory of definite integrals.

Chapter VI is somewhat in the nature of a digression since it comes between a chapter on power series and Chapter VII, the theory of monogenic analytic functions. The last mentioned chapter is based chiefly on lectures of Weierstrass and was not contained in the Allgemeine Arithmetik. It is followed by a chapter on the circular functions which contains in more extended and detailed form the contents of chapter 6, volume II of the older work, *i. e.*, the theory of the functions a^x, $\log x$, $\sin x$, $\sin^{-1}x$, etc., for complex values of x.

Chapters IX, X, and XI have all been revised and enlarged by Gmeiner. The chapter on infinite products (IX) differs from the corresponding chapter of the Allgemeine Arithmetik in that the series formed by taking the logarithms of the factors is used as little as possible. There have been added a number of theorems due to Arzelà about products whose factors are functions of an independent variable, thus involving the notion of uniform convergence. The treatment of finite and infinite continued fractions (X, XI) has also been much extended by Gmeiner. Among other changes, such expressions as

$$3 + \frac{2}{7} + \frac{5}{2} - \frac{1}{1} - \frac{1}{2},$$

which in the Allgemeine Arithmetik were regarded as "sinn-

los"* are here designated as "improper" continued fractions and their value determined by the following definition: The expression

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \tag{1}$$

determines the quantities Z_k by the equations

$$\begin{aligned} Z_n &= b_n, \quad Z_{n+1} = 1, \\ Z_m &= b_m Z_{m+1} + a_{m+1} Z_{m+2} \\ &\qquad (m = n-1, n-2, \cdots, 1, 0). \end{aligned} \tag{2}$$

The value of (1) is Z_0/Z_1. The only fractions which now fail to have values are those for which Z_1 is zero. If $Z_\kappa = 0$ for any value of κ other than 1, the fraction is improper. This definition leads to no new extension of the notion of an infinite continued fraction because improper convergents had been used previously even where the corresponding continued fractions were regarded as meaningless. Among the modern developments added to the chapter on infinite continued fractions, it is interesting to note the theorems of Van Vleck published in the second volume of the *Transactions*.

To sum up this notice and also the review of the first volume in last December's BULLETIN — the work as a whole gives an effect of conservatism, maturity and poise. It is not as likely as the modern French books to stimulate research, but it has a permanent value as a repository of accurate information about the conventional functions and processes of analysis.

OSWALD VEBLEN.

Lezioni sul Calcolo degli Infinitesimi, date nella R. Università di Modena da ETTORE BORTOLOTTI. Raccolte dal Dr. ARMANDO BARBIERI. Modena, 1905. vii + 61 pp.

BORTOLOTTI's Calcolo degli infinitesimi is a short course of lectures on various questions that arise in connection with the determination of the relative orders of two infinitesimals (infinites)

*Because $2 - \frac{1}{1} - \frac{1}{2} = 0$. The significance of this definition of "improper" continued fractions may perhaps be suggested by the equation

$$3 + \frac{2}{7} + \frac{5}{0} = 3 + \frac{2 \cdot 0}{0 \cdot 7 + 5} = 3.$$

$f(x)$ and $\phi(x)$, and consequently includes a discussion of the indeterminate forms $\frac{0}{0}$, $\frac{\infty}{\infty}$, etc. The book is intended for comparatively young students. It begins by defining the upper and lower bounds of indetermination of $\psi(x)$ as x approaches a limiting value and recapitulating some general theorems on limits. It then defines the equality of orders of $f(x)$ and $\phi(x)$ as meaning that $|f(x)/\phi(x)|$ has finite and positive bounds of indetermination. If the limit of $f(x)/\phi(x)$ exists and is infinite as x approaches a limiting value, $f(x)$ is said to be an infinitesimal of lower order than $\phi(x)$; $f(x)$ is of order m if of the same order as $(x-a)^m$.

The difficulties of this definition, for example that functions exist whose order is less than $m+\alpha$ and greater than $m-\alpha$ for every $\alpha>0$ without being equal to m, are exhibited and used to lead up to Cauchy's definition: $f(x)$ is infinitesimal of the same order as $\phi(x)$ if for every constant $\alpha>0$,

$$\lim_{x=a} \frac{f(x)}{[\phi(x)]^{1+\alpha}} = \infty, \quad \lim_{x=a} \frac{f(x)}{[\phi(x)]^{1-\alpha}} = 0.$$

This, as Bortolotti points out, is equivalent to saying that when $\epsilon(x)$ is defined by the equation $f(x)/\phi(x) = [\phi(x)]^{\epsilon(x)}$,

$$\lim_{x=a} \epsilon(x) = 0.$$

This definition is subject to the inconvenience of requiring two functions to be of the same order in some cases where their ratio approaches the limit ∞, but it has the advantage of extending the notion of order to a wider class of functions than is reached by the first definition.

After developing the properties of Cauchy's definition of order, Bortolotti goes on to expound in a clear and rigorous way the rules derived by l'Hôpital, Cauchy and Stolz for the evaluation of the indeterminate forms $0/0$ and ∞/∞. Here, it is to be remarked that to the hypothesis of the theorem on page 25 should be added the sentence, "If $\phi(x)$ is infinitesimal, so is also $f(x)$."

These more or less familiar theorems are followed by methods of determining the relative orders of functions by examining the double ratios

$$\frac{f'(x)}{\phi'(x)} : \frac{f(x)}{\phi(x)}, \quad \frac{\Delta f}{\Delta \phi} : \frac{f}{\phi}.$$

Some of these criteria might perhaps have been stated to better advantage in terms of the derivatives of the logarithms of $f(x)$ and $\phi(x)$. The book ends with a consideration of the indeterminate forms $\infty - \infty$, 1^{∞}, etc.

Taken as a whole, the book is not only clear, readable, and fairly thorough, but has a good set of references to other works and a rich list of problems which may be useful in elementary calculus courses. It therefore seems to deserve a place in any fairly complete mathematical library.

OSWALD VEBLEN.

Abstrakte Geometrie. By KARL THEODOR VAHLEN. Leipzig, B. G. Teubner, 1905. xi + 302 pp.

A book following the general outline of Vahlen's Abstrakte Geometrie would be very useful for giving a general view of the recent studies on foundations of mathematics. This is sufficiently indicated by the titles of the chapters: I, Foundations of arithmetic; II, Projective geometry (theorems of connection); III, Projective geometry (theorems of order); IV, Affine geometry (euclidean and non-euclidean); V, Metric geometry.

Unfortunately, however, the book is not characterized by that precision of language which is indispensable in any discussion of such a subject. The reader is constantly confronted with statements which are incorrect if taken literally and which, if not taken literally, are open to more than one interpretation. Many of the author's postulates are labeled by him as definitions. Moreover, there are places where it is very difficult to determine which of the previously stated hypotheses are being used and which are not. As a consequence, the reviewer is able to state hardly a single new *result* which is surely established by this book.

On the other hand, there are many *suggestions* of methods which if rigorously carried out would probably lead to interesting and elegant results. For example, the notion of planar order is defined not by means of coördinates as in the usual analysis, nor by the way in which a straight line intersects a triangle (according to Pasch) but by means of postulates in terms of the right and left sides of a line with respect to a given sense on the line. One is thus enabled to deal at once with the most general type of planar-ordered set without presupposing anything about a plane in which it lies.

For suggestions such as the one mentioned, the book may be recommended to anyone who is already familiar with other works on the foundations of geometry, but for a beginner it would be thoroughly misleading, and to a philosophical outsider who wished to learn the methods and ideas of the logic of mathematics it would give some very queer notions indeed.

It seems to the reviewer not to be worth while to lengthen this notice with criticisms of details, especially as many of the points that would be mentioned have already been adverted to by Dehn in a review published in the *Jahresbericht der Deutschen Mathematiker-Vereinigung*, volume 14, page 535. The reader who is interested in such things will find a rejoinder to Dehn by Vahlen on page 591 of the same volume, a retort by Dehn on page 595, and a second "Erwiderung" by Vahlen in volume 15, page 73. With these he may compare a footnote by Schoenflies on page 31, volume 15.

OSWALD VEBLEN.

Quadratic Partitions. By Lt. Col. ALLAN CUNNINGHAM, R.E. London, Francis Hodgson, 1904. xxiii + 266 pp.

THE main tables in the book under review contain the quadratic parts (t, u) of the partitions

$$p = t^2 \pm Du^2,$$

wherever possible, for all values of $D < 20$ ($D \neq k^2 \cdot \delta$), and for all primes p to various limits not above 100,000. Shorter tables at the end of the book contain solutions of the Pellian equations

$$\tau^2 \pm Dv^2 = \pm 1, \quad \pm 2, \quad \pm 4, \quad \pm 8, \quad \pm 16$$

for various values of D not over 1,000.

In the Introduction the author gives a brief, but excellent sketch of the properties of quadratic forms, and describes methods of applying the present tables to factorizations, the calculation of Hauptexponenten and kindred problems.

Comparing it with the earlier tables of Jacobi and Reuschle, the student of number theory cannot fail to be impressed with the excellence of Col. Cunningham's book. Neatness, freedom from errors, and admirably compact arrangement of tables of such extent add to their appearance as well as their usefulness. The odd primes are printed *forty* on each page, an arrange-

ment which greatly facilitates counting the primes that fall in a given interval, and reduces the liability to error in computations where a number of successive primes are used.

In addition to errata noted by the author, an obvious misprint occurs in the Introduction: page ix, line 4 from bottom, for $(3\tau^2 v + qv^3)$ read $(3\tau^2 v + Dv^3)^2$.

J. C. MOREHEAD.

ERRATA.

The following errata in the present volume of the BULLETIN have come to the attention of the editors:

Page 74, line 28, *for* k *read* $k > 1$.

Page 75, line 38, *for* $3^{k+\beta}$ *read* $2^{k+\beta}$.

line 42, *for* 0, 1 *read* 0, n; $n > 0$.

NOTES.

AT the meeting of the London mathematical society, held on May 10, the following papers were read: By B. RUSSELL, "On the substitution theory of classes and relations"; by E. CUNNINGHAM, "On linear differential equations of rank unity"; by E. J. ROUTH, "On the motion of a swarm of particles whose center of gravity describes an elliptic orbit of small eccentricity about the sun"; by H. BATEMAN, "The theory of integral equations"; by G. H. HARDY, "Singularities of power series in two variables."

THE appearance of the April number (volume 28, number 2) of the *American Journal of Mathematics* has been delayed by an extensive printers' strike in Baltimore.

THE University of Kiel announces the following prize problem for the year 1906–1907:

It is required to determine the relations which exist between the principal integrals of the various regions, for the hypergeometric differential equation of the third order with two finite singular points.

THE following advanced courses in mathematics are offered during the year 1906–1907 by the universities named below:

JOHNS HOPKINS UNIVERSITY. — By Professor F. MORLEY: Projective geometry, two hours; Dynamics, two hours (first half year); Theory of functions, two hours (second half year); Classic authors, one hour. — By Dr. A. COHEN: Elementary theory of functions, two hours; Differential equations, three hours (first half year); Theory of numbers, three hours (second half year). — By Dr. A. B. COBLE: Theory of correspondence, two hours.

INDIANA UNIVERSITY. — By Professor R. J. ALEY: Differential equations, five hours; Theory of numbers, six hours. — By Professor S. C. DAVISSON: Modern analytic geometry, four hours; Theory of surfaces, four hours; Fourier's series, three hours. — By Professor D. A. ROTHROCK: Calculus, II, six hours; Calculus of variations, six hours; Functions defined by differential equations, four hours. — By Professor U. S. HANNA: Groups of substitutions, three hours; Galois's theory of equations, three hours.

(Summer term, June 21–September 7, 1906). — By Professor S. C. DAVISSON: Higher plane curves, five hours. — By Professor D. A. ROTHROCK: Calculus of variations, six hours. — By Professor U. S. HANNA: Theory of numbers, three hours.

UNIVERSITY OF WISCONSIN (Summer session). — By Professor C. S. SLICHTER: History of mathematics, two hours; Differential equations, five hours. — By Professor G. A. BLISS: Elliptic functions in the Jacobi form, five hours; Calculus of variations, three hours.

PROFESSORS G. CASTELNUOVO and V. CERRUTI, of the University of Rome, and Professor A. CAPELLI, of the University of Naples, have been elected to membership in the royal institute of Lombardy.

DR. P. CALAPSO has been appointed docent of analytic and projective geometry at the University of Palermo. Dr. G. MARLETTA was appointed to a similar position at the University of Catania.

PROFESSOR D. v. STERNECK has been promoted to a full professorship of mathematics at the University of Czernowitz.

DR. A. HASENÖHRL, of the University of Vienna, has been appointed associate professor of physics at the technical school of Vienna.

PROFESSOR N. N. SALTYKOW, of the technical school at Kiew, has been appointed associate professor of pure mathematics at the University of Charkow.

PROFESSOR W. A. STEKLOFF, of the University of Charkow, has been appointed professor of applied mathematics at the University of St. Petersburg.

DR. K. WIEGHARDT, of the technical school at Aachen, has been appointed associate professor of mechanics at the technical school at Brunswick.

DR. E. SCHMID has been appointed docent in mathematics at the University of Bonn, and DR. SCHELLFISCH to a similar position at the University of Münster.

PROFESSOR E. W. BROWN, of Haverford College, has been appointed to a professorship of mathematics at Yale University. Professor Brown will remain at Haverford until the autumn of 1907.

DR. R. B. ALLEN, of Clark University, has been appointed professor of mathematics at Kenyon College, Gambier, Ohio.

AT Columbia University, Dr. EDWARD KASNER has been promoted to an associate professorship of mathematics.

DR. H. F. STECKER has been promoted to an assistant professorship of mathematics at the Pennsylvania State College.

AT Harvard University, Mr. J. K. WHITTEMORE has been promoted to an assistant professorship of mathematics.

MR. A. S. EVE has been appointed assistant professor of mathematics in McGill University.

MR. F. R. SHARPE, Dr. ARTHUR RANUM, and Dr. W. B. CARVER have been appointed instructors in mathematics at Cornell University.

DR. R. L. MOORE has been appointed instructor in mathematics at Princeton University.

MR. W. M. THOMAS has been appointed tutor in mathematics in the College of the City of New York.

MISS M. M. YOUNG has been appointed instructor in mathematics in Wellesley College.

DR. ROXANA H. VIVIAN, instructor in mathematics at Wellesley College, will be a member of the faculty of the American college for girls in Constantinople the coming year, on leave of absence from Wellesley.

PROFESSOR GUSTAV BAUER, of the University of Munich, died April 3, at the age of 86 years.

PROFESSOR G. OLTRAMARE, emeritus professor of the University of Geneva, died April 10 at the age of 90 years.

PROFESSOR G. A. WENTWORTH, formerly of Phillips Exeter Academy, died May 24, at the age of 71 years. Professor Wentworth had been a member of the AMERICAN MATHEMATICAL SOCIETY since 1891.

Catalogues of second hand mathematical works: Gustav Fock, 7 Schlossgasse, Leipzig, catalogue no. 280, 2802 titles in mathematics, astronomy, geology; W. Junk, 201 Kurfürstendamm, Berlin, catalogue no. 28, about 2400 titles in astronomy, physics, and mathematics.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

CLEBSCH (A.). Vorlesungen über Geometrie, bearbeitet und herausgegeben von F. Lindemann. (In 2 Bänden.) 1ter Band: Geometrie der Ebene. 1ter Teil: Kegelschnitte und algebraische Formen. 1te Lieferung. 2te, vermehrte Auflage. Leipzig, Teubner, 1906. 8vo. 480 pp. M. 12.00

FABER (G.). Ueber die zusammengehörigen Konvergenzradien von Potenzreihen mehrerer Veränderlicher. Karlsruhe, 1905. 8vo. 36 pp.

FISHER (I.). A brief introduction to the infinitesimal calculus; designed especially to aid in reading mathematical economics and statistics. New edition. New York, Macmillan, 1906. 12mo. 13 + 84 pp. Cloth. $0.75

GARBIERI (G.). Complementi di algebra: riassunto di lezioni date nelle università di Padova e di Genova. Torino, Paravia, 1906. 8vo. 32 pp. L. 2.00

HAGEN (J. G.). Synopsis der höheren Mathematik. Band III: Differential- und Integralrechnung. 7te Lieferung. Berlin, Dames, 1905. 4to. 6 + pp. 385–471. M. 5.00

HESS (A.). Stetige Abbildung einer Linie auf ein Quadrat. Zürich, 1905. 8vo. 45 pp. M. 2.00

KLEIN (F.). Autographierte Vorlesungshefte. Heft III: Ueber die hypergeometrische Funktion. Vorlesung, gehalten im Wintersemester 1893–94. Ausgearbeitet von E. Ritter. Neuer unveränderter Abdruck. 4 + 568 pp. M. 9.00

——. Heft V: Riemannsche Flächen. Ite Vorlesung, gehalten während des Wintersemesters 1891–92. Neuer unveränderter Abdruck. 4 + 301 pp. IIte Vorlesung, gehalten während des Sommersemesters 1892. Neuer unveränderter Abdruck. 4 + 288 pp. Leipzig, Teubner, 1906. M. 1 .00

LEBESGUE (H.). Leçons sur les séries trigonométriques, professées au Collège de France. Paris, Gauthier-Villars, 1906. 8vo. 136 pp. Fr. 3.50

LEBON (E.). Tables de caractéristiques relatives à la base 2,310 des facteurs premiers d'un nombre inférieur à 30,030. Paris, Delalain, 1906. 8vo. 32 pp. Fr. 1.50

LÉVY. See ROUCHÉ.

LINDEMANN (F.). See CLEBSCH (A.).

LINDOW (M.). See SOHNCKE (L. A.).

MANNING (H. P.). Irrational numbers and their representation by sequences and series. New York, Wiley, 1906. 12mo. 6 + 123 pp. Cloth. $1.25

MATTSON (R.). Contributions à la théorie des fonctions entières. Upsala, 1905. 8vo. 97 pp. M. 2.50

RITTER (E.). See KLEIN (F.).

ROUCHÉ et LÉVY. Analyse infinitésimale, à l'usage des ingénieurs. Tome 2. Paris, Gauthier-Villars, 1905. 8vo. 7 + 648 pp. Fr. 15.00

SOHNCKE (L. A.). Sammlung von Aufgaben aus der Differential- und Integralrechnung. Teil II. 2te Abteilung: Integralrechnung. II. 6te verbesserte Auflage. Bearbeitet und herausgegeben von M. Lindow. Jena, Schmidt, 1906. 8vo. 6 + 224 pp. M. 5.00

VALLÉE POUSSIN (C. J. DE LA). Cours d'analyse infinitésimale. Tome II. Louvain, 1906. 8vo. 16 + 440 pp. Fr. 12.00

VERONESE (G.). Il vero nella matematica. Roma, Forzani, 1906. 8vo. 38 pp.

II. ELEMENTARY MATHEMATICS.

BLOMFIELD (C. H.). See JONES (A. C.).

CRACKNELL (A. G.). See WORKMAN (W. P.).

DEAKIN (R.). Euclid, Books I, III. Preliminary certificate edition (for course B). London, Clive, 1906. 8vo. 296 pp. Cloth. 2s. 6d.

D'Ovidio (E.). See Sannia (A.).

Eggar (W. D.). Manual of geometry. London, Macmillan, 1906. 8vo. 350 pp. Cloth. 3s. 6d.

Fernández de Prado (G.). Manual de la regla de cálculo. Exposición teórico-práctica con aplicación á las cuestiones más usuales de aritmética, algebra, geometría, trigonometría y numerosos ejercicios y problemas, etc. Madrid, Asilo, 1906. 137 pp. P. 3.00

Grévy (A.). Géométrie dans l'espace, à l'usage des élèves des classes de première C et D (programme du 27 juillet 1905). Paris, Vuibert, 1906. 8vo. 214 pp. Fr. 2.00

Hartenstein. Resultate und Auflösungen zu "Bardey-Hartenstein, Arithmetische Aufgaben nebst Lehrbuch der Arithmetik, 6te Auflage und folgende." Leipzig, Teubner, 1906. 8vo. 80 pp. M. 1.80

Jones (A. C.) and Blomfield (C. H.). Test papers in elementary mathematics. London, Arnold, 1906. 8vo. 256 pp. 2s. 6d.

Kambly und Roeder. Stereometrie und sphärische Trigonometrie. Nach den preussischen Lehrplänen von 1901 umgearbeitet. Ausgabe der Stereometrie und der sphärischen Trigonometrie von Kambly. Lehraufgabe der Prima. Mit Uebungsaufgaben und einem Anhang: Der Koordinatenbegriff und einige Grundeigenschaften der Kegelschnitte. 4te, durchgesehene Auflage. (31te der Kamblyschen Stereometrie). Breslau, Hirt, 1906. 8vo. 224 pp. M. 2.30

Krebs (A.). Allgemeine Arithmetik in elementarer Darstellung, für den Mittelschul- und Selbstunterricht in entwickelnder Lehrform bearbeitet. 2tes Heft. Bern, Baumgart, 1906. 8vo. 3 + pp. 81–147. M. 1.30

London University matriculation mathematics papers. From Jan. 1896 to Jan. 1906. (*University Tutorial Series.*) London, Clive, 1906. 8vo. 120 pp. 1s. 6d.

Marks (C. I.). Mathematical questions and solutions. From the "Educational Times." New series. Vol. 9. London, Hodgson, 1906. 8vo. Cloth. 6s. 6d.

Maurer (H.). Methodisch geordnete Sammlung geometrischer Aufgaben in bildlicher Darstellung. 3360 Aufgaben in 4 Bänden. Zum Selbststudium und zum Unterricht an höheren Lehranstalten. 1ter Band (Aufgaben 1–840). Zürich, Speidel, 1906. 8vo. M. 2.50

M. (F. G.). Compléments de trigonométrie et méthodes pour la résolution des problèmes. Paris, Poussielgue, 1906. 8vo. 16 + 800 pp.

Müller (H.) und Plath (J.). Ergebnisse zu der Aufgabensammlung. Leipzig, Teubner, 1906. 8vo. 60 pp. M. 1.40

Plath (J). See Müller (H.).

Reinhard (A. I.). High school algebra. With answers. New York, American Book Co., 1906. 12mo. 422 pp. Cloth. $1.00

Robbins (E. R.) and Somerville (F. H.). Exercises in algebra. New York, American Book Co., 1906. 12mo. 173 pp. Half leather. $0.50

——. Answers to exercises in algebra. New York, American Book Co., 1906. $0.10

ROEDER. See KAMBLY.

SANDERS (A.). Plane and solid geometry. New York, American Book Co., 1906. $1.25

SANNIA (A.) e D'OVIDIO (E.). Elementi di geometria. Volume 1, ad uso dei ginnasii. 12a edizione. Napoli, 1906. 8vo. 16 + 200 pp. L. 2.00

SCOTTI (G.). Elementi di geometria intuitiva ad uso del ginnasio inferiore e dei corsi complementari. 8a edizione. Torino, Salesiana, 1906. 16mo. 139 pp. L. 1.00

SOMERVILLE (F. H.). See ROBBINS (E. R.).

WENTWORTH (G. A.). Elementary algebra. Boston, Ginn, 1906. 12mo. 10 + 421 pp. Half leather. $1.12

WORKMAN (W. P.) and CRACKNELL (A. G.). Geometry. Preliminary certificate edition. For course A. London, Clive, 1906. 8vo. 380 pp. Cloth. 3s. 6d.

III. APPLIED MATHEMATICS.

CALDARERA (F.). Corso di meccanica razionale. Volume III: Equilibrio e moto dei sistemi continui; idrostatica; idrodinamica. Palermo, 1906. 8vo. 268 pp. L. 11.00

CONSTAN (P.). Tables graphiques d'azimut. Paris, Gauthier-Villars, 1906. 4to. 50 pp. Fr. 3.00

DALBY (W. E.). Balancing of engines. 2d edition. London, Arnold, 1906. 8vo. 316 pp. Cloth. 10s. 6d.

FERRIS (C. E.). Elements of descriptive geometry. New York, American Book Co., 1906. 8vo. 135 pp. Cloth. $1.25

FLEMING (J. A.). Principles of electric wave telegraphy. London, Longmans, 1906. 8vo. 692 pp. Cloth. 24s.

GALILEI (G.). Le opere. Edizione nazionale sotto gli auspicî di Sua Maestà il Re d'Italia. Volume XVII. Firenze, Barbèra, 1906. 4to. 438 pp.

HOUSUM (C.). See TRINKS (W.).

KÜTTNER (W.). Das Risiko der Lebensversicherungsanstalten und Unterstützungskassen. Berlin, 1906. 8vo. 95 pp. M. 4.00

MARTIN (L. A.). Textbook of mechanics. Volume 1: Statics. New York, Wiley, 1906. 12mo. 12 + 142 pp. Cloth. $1.25

MORGAN (J. L. R.). Physical chemistry for electrical engineers. New York, Wiley, 1906. 12mo. 8 + 230 pp. Cloth. $1.50

MOULTON (F. R.). An introduction to astronomy. New York, Macmillan, 1906. 8vo. 18 + 557 pp. Cloth. $1.25

ORLICH (E.). Aufnahme und Analyse der Wechselstromkurven. Braunschweig, 1906. 8vo. 8 + 117 pp. M. 3.50

PAUSER (A.). Die Elektrizität als Bewegung. (Progr.) Villach, 1905. 8vo. 21 pp.

PETROVITCH (M.). La mécanique des phénomènes fondée sur les analogies. Paris, Gauthier-Villars, 1906. 8vo. 96 pp. Fr. 2.00

RIBERA Y URUBURU (L. DE). Tratado de náutica, adecuado para el estudio de dicha ciencia en la marina mercante, 1a y 2a parti in 1 vol. Ferrol, El Correo Gallego, 1906. 673 pp. P. 14.00

RICHARD (G.). Mécanique. A l'usage des ingénieurs, constructeurs mécaniciens, industriels, chefs d'ateliers et contremaîtres. 28e édition, revue et augmentée. Paris, Dunod, 1906. 16mo. 5 + 205 + 64 pp. Fr. 2.50

SÁNCHEZ PASTORFIDO (D. C.). Resistencia de materiales y sus aplicaciones á la construcción de máquinas. 2a edición. Madrid, Arias, 1906. 331 pp. P. 14.00

TRINKS (W.) and HOUSUM (C.). Shaft governors. New York, Van Nostrand, 1906. 16mo. 97 pp. $0.50

FIFTEENTH ANNUAL LIST OF PAPERS

READ BEFORE THE AMERICAN MATHEMATICAL SOCIETY AND SUBSEQUENTLY PUBLISHED, INCLUDING REFERENCES TO THE PLACES OF THEIR PUBLICATION.

AMES, L. D. Supplementary Communication on the Division of Space by a Closed Surface. Read Sept. 16, 1904. *American Journal of Mathematics*, vol. 27, No. 4, pp. 343–380; Oct., 1905.

BIRKHOFF, G. D. General Mean Value and Remainder Theorems with Applications to Mechanical Differentiation and Quadrature. Read Feb. 27, 1904. *Transactions of the American Mathematical Society*, vol. 7, No. 1, pp. 107–136; Jan., 1906.

BLISS, G. A. A Generalization of the Notion of Angle. Read Sept. 7, 1905. *Transactions of the American Mathematical Society*, vol. 7, No. 2, pp. 184–196; Apr., 1906.

—— A Proof of the Fundamental Théorem of Analysis Situs. Read Dec. 28, 1905. *Bulletin of the American Mathematical Society*, vol. 12, No. 7, pp. 336–341; Apr., 1906.

BLISS, G. A. and MASON, MAX. A Problem of the Calculus of Variations in which the Integrand is Discontinuous. Read Oct. 28, 1905. *Transactions of the American Mathematical Society*, vol. 7, No. 2, pp. 325–336; Apr., 1906.

BOLZA, OSKAR. A Fifth Necessary Condition for a Strong Extremum of the Integral

$$\int_{x_0}^{x_1} F(x, y, y')dx.$$

Read Feb. 24, 1906. *Transactions of the American Mathematical Society*, vol. 7, No. 2, pp. 314–324; Apr., 1906.

BROMWICH, T. J. I'A. The Classification of Quadrics. Read Sept. 17, 1904. *Transactions of the American Mathematical Society*, vol. 6, No. 3, pp. 275–285; July, 1905.

BROWN, E. W. On a General Method for Treating Transmitted Motions and its Application to Indirect Perturbations. Read Feb. 25, 1905. *Transactions of the American Mathematical Society*, vol. 6, No. 3, pp. 332–343; July, 1905.

BUSSEY, W. H. Galois Field Tables for $p^n \leqq 169$. Read Sept. 7, 1905. *Bulletin of the American Mathematical Society*, vol. 12, No. 1, pp. 22–38; Oct., 1905.

—— See VEBLEN, OSWALD.

CARSTENS, MISS R. L. A Definition of Quaternions by Independent Postulates. Read Feb. 24, 1906. *Bulletin of the American Mathematical Society*, vol. 12, No. 8, pp. 392–394; May, 1906.

CARVER, W. B. On the Cayley-Veronese Class of Configurations. Read Oct. 28, 1905. *Transactions of the American Mathematical Society*, vol. 6, No. 4, pp. 534–545; Oct., 1905.

COBLE, A. B. On the Relation between the Three-Parameter Groups of a Cubic Space Curve and a Quadric Surface. Read Dec. 29, 1904. *Transactions of the American Mathematical Society*, vol. 7, No. 1, pp. 1–20; Jan., 1906.

CURTISS, D. R. Sur la Théorie des Fonctions hypergéométriques. Read Sept. 16, 1904. *Annales Scientifiques de l'Ecole Normale Supérieure*, ser. 3, vol. 22, No. 3, pp. 121–143; Mar., 1906.

—— Theorems Converse to Riemann's on Linear Differential Equations. Read Apr. 29, 1905. *Transactions of the American Mathematical Society*, vol. 7, No. 1, pp. 99–106; Jan., 1906.

—— On Certain Properties of Wronskians and Related Matrices. Read (Chicago) Apr. 14, 1906. *Bulletin of the American Mathematical Society*, vol. 12, No. 10, pp. 482–485; July, 1906.

DAVIS, E. W. The Elliptic Functions and the General Symmetric Group in Four Letters. Read (Chicago) Dec. 31, 1903. *Nebraska University Studies*, vol. 4, No. 3, pp. 231–247; July, 1904.

DICKSON, L. E. Subgroups of Order a Power of p in the General and Special m-ary Linear Homogeneous Groups in the $GF[p^n]$. Read Sept. 16, 1904. *American Journal of Mathematics*, vol. 27, No. 3, pp. 280–302; July, 1905.

—— On Hypercomplex Number Systems. Read (Chicago) Apr. 22, 1905. *Transactions of the American Mathematical Society*, vol. 6, No. 3, pp. 344–348; July, 1905.

—— On the Real Elements of Certain Classes of Geometrical Configurations. Read (Chicago) Apr. 22, 1905. *Annals of Mathematics*, ser. 2, vol. 6, No. 4, pp. 141–150; July, 1905.

—— On Finite Algebras. Read Sept. 8, 1905. *Göttinger Nachrichten*, 1905, No. 4, pp. 358–393.

—— On the Quaternary Linear Homogeneous Groups Modulo p of Order a Multiple of p. Read Sept. 8, 1905. *American Journal of Mathematics*, vol. 28, No. 1, pp. 1–16; Jan., 1906.

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